To find the condition that the ellipsoid shall be tangent to conical surface, we assume the equation of complete cone to be:

$$m^2(x^2+z^2)=(y-k)^2 \ldots\ldots \ldots\ldots\ldots\ldots \ldots\ldots (5).$$

For intersection of (1) and (5),

$$(A-H)m^2x^2+(Bm^2+H)y^2+Cm^2xy+Dm^2x+(Em^2-2k)+(Hk^2+Fm^2)=0.$$

If this ellipse have no axes,

$$(Hk^2+Fm^2)[c^2m^2-4(A-H)(Bm^2+H)]+(A-H)(Em^2-2k)^2+(Bm^2+H)D^2m^2$$
$$-CD(Em^2-2k)m^2=0.$$

Solving this for B we obtain,

$$B=\frac{CD(Em^2-2k)m^2-(A-H)(Em^2-2k)^2-C^2m^2(Hk^2+Fm^2)}{m^2[D^2m^2-4(A-H)(Hk^2+Fm^2)]}-\frac{H}{m^2}.$$

Substitute the value of A given in (3),

$$B=\frac{4FCD(Em^2-2k)m^2-(D^2-4FH)(Em^2-2k)^2-4FC^2m^2(Hk^2+Fm^2)}{m^2[FD^2m^2-(D^2-4FH)(Hk^2+Fm^2)]}-\frac{H}{m^2}.$$

If we were then to substitute these values of A and B in equation (2), we should obtain a value of V which contains the variables C, D, E, F, and H, independent, except as to the condition given in (4). By the ordinary methods of maximum and minimum, five equations can be formed and the maximum critical values of the five letters determined. But life is too short to do this.

If we assume that two of the axes are parallel to the bases of the frustum, we obtain $V=\frac{4}{3}\pi\tan^2\phi(h^2b-2hb^2)$, where h=altitude of complete cone, ϕ=semi-angle of cone, and b=semi-vertical axis of ellipsoid. From this for maximum, $b=p/4$.

52. Proposed by O. W. ANTHONY, M. Sc., Professor of Mathematics and Astronomy, New Windsor College, New Windsor, Maryland.

There are two lights of intensities m and n. Where must a target, whose surface is parallel to the line joining the two lights, be set up in order that it shall receive the maximum illumination per unit of area?

I. Solution by the PROPOSER.

If we take the point where the light with intensity l is situated as the origin of coordinates, we have readily from the principles of Optics,
$I=ly/(x^2+y^2)^{\frac{3}{2}}+my/[(a-x)^2+y^2]^{\frac{3}{2}}$, x and y being the coordinates of the bull's-eye.

$$\frac{dI}{dx}=\frac{-3lxy}{(x^2+y^2)^{\frac{5}{2}}}+\frac{3m(a-x)y}{[(a-x)^2+y^2]^{\frac{5}{2}}}=0\ldots\ldots\ldots\ldots\ldots\ldots(1).$$

$$\frac{dI}{dy}=\frac{l(x^2-2y^2)}{(x^2+y^2)^{\frac{5}{2}}}=\frac{m[(a-x)^2-2y^2]}{[(a-x)^2+y^2]^{\frac{5}{2}}}=0\ldots\ldots\ldots\ldots\ldots\ldots(2).$$

From (1) $y=0$ .... (a),

$$\text{or }\frac{[(a-x)^2+y^2]^{\frac{5}{2}}}{[x^2+y^2]^{\frac{5}{2}}}=\frac{m}{l}\cdot\frac{a-x}{x}\ldots\ldots\ldots\ldots\ldots\ldots(b).$$

$$\text{From (2) }\frac{[(a-y)^2+y^2]^{\frac{5}{2}}}{[x^2+y^2]^{\frac{5}{2}}}=-\frac{m}{l}\cdot\frac{(a-x)^2-2y^2}{x^2-2y^2}\ldots\ldots\ldots\ldots\ldots\ldots(c).$$

From (b) and (c) $y^2=x(a-x)/2$ .... (d).

$$\text{From }(b)\ y=\pm x^{\frac{1}{5}}(a-x)^{\frac{1}{5}}\sqrt{\frac{m^{\frac{2}{5}}x^{\frac{7}{5}}-l^{\frac{2}{5}}(a-x)^{\frac{7}{5}}}{l^{\frac{2}{5}}x^{\frac{2}{5}}-m^{\frac{2}{5}}(a-x)^{\frac{2}{5}}}}\ldots\ldots\ldots\ldots\ldots\ldots(e).$$

By (a) and (d); $x=0$; $x=a$; that is, the lights themselves must be used, as bull's-eye. By (a) and (e) we obtain the additional condition $x=al^{\frac{1}{3}}/(l^{\frac{1}{3}}+m^{\frac{1}{3}})$, which is the point of minimum illumination on the line joining the two lights. Other critical points will be obtained by solving (d) and (e) simultaneously,—a task which seems to be almost impossible.

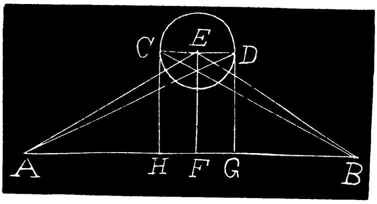
II. Solution by G. B. M. ZERR, A. M., Ph. D., Texarkana, Arkansas-Texas.

Let A, B, be the lights, intensities m, n; E the center of the target, radius $ED=r$, $AB=a$, $AF=x$, $EF=z$, $\angle DAB=\theta$, $\angle CBA=\phi$.

$\therefore\ m\sin\theta/AD^2+n\sin\phi/BC^2=I.$

$\sin\theta=z/AD=z/\sqrt{z^2+(x+r)^2}$,

$\sin\phi=z/\sqrt{z^2+(a-x+r)^2}$.



$$\therefore\frac{mz}{\{z^2+(x+r)^2\}^{\frac{3}{2}}}+\frac{nz}{\{z^2+(a-x+r)^2\}^{\frac{3}{2}}}=I.$$

Differentiating with reference to z,

$$\frac{m(x+r)^2-2mz^2}{\{z^2+(x+r)^2\}^{\frac{5}{2}}}+\frac{n(a-x+r)^2-2nz^2}{\{z^2+(a-x+r)^2\}^{\frac{5}{2}}}=0\ldots\ldots\ldots\ldots(1).$$

Differentiating with respect to x,

$$\frac{m(x+r)}{\{z^2+(x+r)^2\}^{\frac{5}{2}}}=\frac{n(a-x+r)}{\{z^2+(a-x+r)^2\}^{\frac{5}{2}}}\ldots\ldots\ldots\ldots(2).$$

From (1) and (2), $z=\sqrt{\tfrac{1}{2}(x+r)(a-x+r)}$. This value of z in (2) gives

$$\frac{m(x+r)}{\{(x+r)(a+x+3r)\}^{\frac{3}{2}}}=\frac{n(a-x+r)}{\{(a-x+r)(2a+3r-x)\}^{\frac{3}{2}}},$$

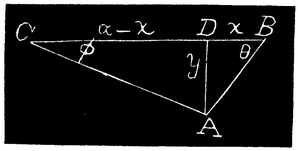
an equation of the eighth degree to find x.

If $m=n$, $x=\tfrac{1}{2}a$, $z=(\tfrac{1}{2}a+r)\sqrt{\tfrac{1}{2}}=\tfrac{1}{2}(\tfrac{1}{2}a+r)\sqrt{2}$.

If $n=0$, $x=0$, $z=\tfrac{1}{2}r\sqrt{2}$.

III. Solution by C. W. M. BLACK, A. M., Professor of Mathematics in Wesleyan Academy, Wilbraham, Massachusetts.

Let A be any position of target, $AD(=y)$ be perpendicular from A to BC, the line connecting the positions of the two lights. Let x equal part of $BC(=a)$ cut off by AD. By laws of light, intensity of light received from B at A



$$=\frac{m\sin\theta}{AB^2}=\frac{m}{x^2+y^2}\times\frac{y}{\sqrt{x^2+y^2}}=\frac{my}{(x^2+y^2)^{\frac{3}{2}}}.$$

Similarly, that received from $C=\dfrac{ny}{[(a-x)^2+y^2]^{\frac{3}{2}}}$.

Then total intensity at A or $n=my(x^2+y^2)^{-\frac{3}{2}}+ny[(a-x)^2+y^2]^{-\frac{3}{2}}$. ..(1).

$$du/dx=-3mxy(x^2+y^2)^{-\frac{5}{2}}+3n(a-x)y[(a-x)^2+y^2]^{-\frac{5}{2}}\text{..........(2).}$$

$$du/dy=m(x^2+y^2)^{-\frac{3}{2}}-3my^2(x^2+y^2)^{-\frac{5}{2}}+n[(a-x)^2+y^2]^{-\frac{3}{2}}-3ny^2[(a-x)^2+y^2]^{-\frac{5}{2}}\text{........ ......(3).}$$

Equating (3) to 0, we have

$$\left.\begin{array}{l} y=0\text{............ ... .(4).} \\ \dfrac{[(a-x)^2+y^2]^{\frac{5}{2}}}{(x^2+y^2)^{\frac{5}{2}}}=\dfrac{n(a-x)}{mx}\text{................ (5).}\end{array}\right\}$$

Equating (3) to 0, we have

$$[(a-x)^2+y^2]^{-\frac{5}{2}}\{3ny^2-n[(a-x)^2+y^2]\}=(x^2+y^2)^{-\frac{5}{2}}[m(x^2+y^2)-3my^2],$$

$$\text{or }\frac{[(a-x)^2+y^2]^{\frac{5}{2}}}{(x^2+y^2)^{\frac{5}{2}}}=\frac{n[2y^2-(a-x)^2]}{m(x^2-2y^2)}\text{ ....(6).}$$

Solving (4) and (6), $\dfrac{(a-x)^5}{x^5}=\dfrac{-n(a-x)^2}{mx^2}$, which gives

$$\left\{\begin{array}{l} x=a \text{ or } 0 \text{ or } \dfrac{a}{1-\sqrt[3]{m\div n}} \\ \text{and } y=0, \end{array}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots(7).$$

From (5) and (6) $\dfrac{n(a-x)}{mx}=\dfrac{n[2y^2-(a-x)^2]}{m(x^2-2y^2)}$, and $y^2=\dfrac{x(a-x)}{2}$......(8)

Instead of finding second differential coefficients, substitute from (7) in (1), $x=a$ and $x=0$, make $n=\infty$. $x=\dfrac{a}{1-\sqrt[3]{n\div m}}$, makes $n=0$.

We can show that (8) does not produce any new condition for a maximum. To make y real x is not <0 nor $>a$.

If $x=a$ or 0, we have the values found in (7). Now for any value of x between 0 and a, y in (8) is seen to be finite, and n in (1) is also finite.

So $x=a$ or $x=0$ with $y=0$, producing the only infinite values of n indicate the positions of maximum intensity of illumination to be directly in front of either light.

PROBLEMS.

59. Proposed by MOSES C. STEVENS, M. A., Department of Mathematics, Purdue University, Lafayette, Indiana.

Solve $n\dfrac{d^2y}{dx^2}(x^2+y^2)^{\frac{1}{2}}=\left[1+\left(\dfrac{dy}{dx}\right)^2\right]^{\frac{3}{2}}$.

[*From Forsyth's Differential Equations.*]

60. Proposed by SETH PRATT, C. E., Assyria, Michigan.

To remove $(1/a)$th of the volume of a sphere of a given radius by a conical hole, whose axis is the axis of a sphere, and whose vertex is at the surface of the sphere. Required the height of the cone and the diameter of its base.

AVERAGE AND PROBABILITY.

Conducted by B. F. FINKEL, Springfield, Mo. All contributions to this department should be sent to him.

NOTE ON PROBLEM 26.

After carefully reading Dr. Martin's "Reply to Replies on Problem 26," we see no reason for changing our opinion respecting the solution we have been defending. We may, however, be led to agree with Dr. E. H. Moore, Dr. William